ii - Year - iii Semester

Course Code : 18ITA3.

Allied Course - iii - Discrete Mathematics

Unit - I

logic : If statements - connective - disjunction - Negation - Conditional statements - Biconditional statements - Atomic and Compound statements - well formed formulae - The truth table - Tautology - Tautological implication formulae with distinct truth tables.

Unit - II

Normal forms : principles of normal forms - theory of inference - open statements - Quantifiers - Valid formulae and equivalence - Theory of inference for predicate calculus.

Unit - III.

Graph theory : Definition - Degrees - Sub graph - Isomorphism - Complete graph - Bipartite graph - paths, cycles - Connectedness.

Unit - IV.

Trees : spanning tree - Kruskal's Algorithm - Prim's Algorithm - Dijkstra's Algorithm - Cut set and cut vertices - eulerian - Hamiltonian graph.

Unit-V

Lattice : Binary relation in a set - partition and covering of a set - equivalance relation - partial ordering - posets - Hasse diagram - lattices - sub lattices properties of sub lattices - special lattice - Boolean Algebra - boolean functions.

Unit-1 (Logic)

Defn:

(SP-Statement of proposition)

A SP-statement of proposition is a declarative sentence to which it is meaningful to assign a truth value 'true' or 'false' but not both simultaneously.

Connectives

Certain keywords, 'and', 'or', 'not', 'if... then', 'if and only if', which are called sentential connectives.

1. Conjunction

The process of joining two statements parody 'and' produces a new statement, denoted by $p \wedge q$ which has the truth value T whenever both p and q have truth value T, otherwise the truth value F. The statement $p \wedge q$ is called the conjunction of the statement p and q.

The truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex: The conjunction of ...

p: Today is sunday

q: Government offices are working

$p \wedge q$: It is sunday and Government offices are working.

2. Disjunction

The process of joining two statements p and q by 'or' produces a new statement. The new statement is disjunction of p and q and denoted by $p \vee q$.

The truth table for $p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: The disjunction of the statements

p: I will by a car

q: I will by a TV.

$p \vee q$: I will by a car or I will by a TV

The statement $p \vee q$ has the truth value 'True' if the truth values of at least one of p and q is 'true'. The truth value of $p \vee q$ is 'false' only when the truth values of both p and q are false.

3. Negation.

The Negation of a statement p is the statement obtained from p by prefixing the words 'It is not truth that'. The negation of p is denoted by $\neg p$.

The truth value of $\neg p$

p	$\neg p$
T	F
F	T

4. Conditional Statements

Let p and q be two given statements. The statement 'if p then q' denoted by $p \rightarrow q$.

The statement $p \rightarrow q$ is called a conditional statement or implication. The statement p is called antecedent, q is called Consequent.

If p and q are two statement, the

statement $p \to q$ has the truth value 'false' when q has the truth value 'false' and p has the truth value 'true'. otherwise $p \to q$ has the truth value 'true'.

The truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

6. Biconditional Statements.

If p and Q are any two statements then the statement $p \leftrightarrow q$ is called a biconditional statement. The statement $p \leftrightarrow q$ has the truth value T whenever both p and q have identical truth values, and in all other cases, the truth value of $p \leftrightarrow q$ is 'false'.

The truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F

Atomic and Compound statements

Statements which do not contain any Connectives are called atomic or primary or simple statements. On the other hand the statements which contain one or more primary statements and at least one Connective are called molecular or Composite ~~one~~ or Compound Statements.

Examples for atomic statements.

1. It is raining.
2. 19 is a positive number.

Examples for Compound Statements

1. Jack and ~~Jill~~ Jill went up the hill
2. It is not true that $3+4=17$

Defn (Statement formula)

A Statement formula is an expression which is a string consisting of Variables (letters - either all of them are Capital or all of them are lower case letters. with or without subscripts, paranthesis, and Connectives symbols ($\wedge, \vee, \rightarrow, \leftrightarrow, \neg$) which produces a statment when the Variables are replaced by statements.

Defn:

A well-formed formula can be generated by the following rules:

1. A statement variable standing alone is a well formed formula.

2. If ϕ is well formed formula, then so is $(\neg\phi)$.

3. If ϕ and ψ are well-formed formula then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are well formed formulae.

4. A string of symbols containing the statement variables, connectives and parantheses is a well-formed formula, if and only if it can be obtained by applying the rules (1), (2) and (3) finitely many times.

The following examples of formulae:

1. $(p_1 \wedge p_2)$ $(\neg(p \wedge q)) \rightarrow (r \wedge (\neg s))$
2. $(\neg((\neg p)) \wedge (\neg q))$

The following examples of not formulae.

1. $((p \rightarrow q) \rightarrow (\wedge q))$
2. $(p \rightarrow q) \rightarrow (\vee p)$

Defn (Sentential Interpretation)

A statement p is said to be a sentential interpretation of a formula $\phi(p_1, p_2 \ldots p_n)$, such that when the variables $p_1, p_2 \ldots p_n$ are replaced by the statements $s_1, s_2, \ldots s_n$ respectively in the formula $\phi(p_1, p_2, \ldots p_n)$ then the resulting statements $\phi(s_1, s_2, \ldots s_n)$ is a sentential interpretation

pb:1 Determine the truth table of the formula $p \rightarrow (Q \rightarrow R)$

P, Q and R are the statement variable that occur in this formula $\phi : p \rightarrow (Q \rightarrow R)$. There are $2^3 = 8$ different sets of truth value assignments for the variables P, Q and R. They are

T	T	T
T	T	F

T F T
T F F
F T T
F T F
F F T
F F F

This 1st row of the truth table of gives the truth value of $s_1 \rightarrow (s_2 \rightarrow s_3)$, where the statements have the s_1, s_2, s_3 truth value 'true', the 2nd row gives the truth value of $s_1 \rightarrow (s_2 \rightarrow s_3)$, where s_1, s_2 and s_3 are construct the remaining rows of the truth table.

the truth table is given below for $p \rightarrow (Q \rightarrow R)$

P	Q	R	$Q \rightarrow R$	$p \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	F	F	F	F
T	T	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

5. Construct the truth table for $\neg(\neg P \wedge \neg Q)$

The Variables that occur in the formula are P and Q so, we have to consider $2^2 = 4$ possible combination of truth values of two statements that replace P and Q

The truth table for $\neg(\neg P \wedge \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

The entries in the last column are the truth values of the formula $\neg(\neg P \wedge \neg Q)$

3. Construct the truth table of the formula $(\neg P \vee Q) \wedge (\neg Q \vee P)$

The Variables that occur in this formula are P and Q. There are 2^2 rows in the truth table of the formula

The truth table for

$(\neg P \vee Q) \wedge (\neg Q \vee P)$

P	Q	¬P	¬Q	¬P∨Q	¬Q∨P	(¬P∨Q)∧(¬Q∨P)
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

4 Construct the truth table for the formula $(p\wedge Q)\vee(\neg p\wedge Q)\vee(p\wedge\neg Q)\vee(\neg p\wedge\neg Q)$

→

Truth table for $(p\wedge Q)\vee(\neg p\wedge Q)\vee(p\wedge\neg Q)\vee(\neg p\wedge\neg Q)$

p	Q	¬p	¬Q	p∧Q	¬p∧Q	p∧¬Q	¬p∧¬Q	(p∧Q)∨(¬p∧Q)∨(p∧¬Q)∨(¬p∧¬Q)
T	T	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	F	T
F	F	T	T	F	F	F	F	T

Tautology

Defn (Tautology)

Let $\phi(p_1, p_2, \ldots p_n)$ be a formula. Then ϕ is said to be a tautology if whenever the variables $p_1, p_2, \ldots p_n$ are replaced by definite statements $s_1, s_2, \ldots s_n$ respectively the resulting statement $\phi(s_1, s_2, \ldots s_n)$ has the truth value 'True'.

Defn (Contradiction)

ϕ is said to be a contradiction if every sentential interpration of ϕ has the truth value 'false'.

Defn (Satisfiable)

ϕ is said to be satisfiable if there at least one sentential interpretation of ϕ which has the truth table value true.

Pr: verify whether $(p \vee Q) \rightarrow p$ is a tautology

$\rightarrow$ Ist we Contrsuct truth table for the formula $(p \vee Q) \rightarrow p$.

Truth table for $(q \vee r) \rightarrow (p \wedge \neg r)$

p	q	r	$q \vee r$	$\neg r$	$p \wedge \neg r$	$(q \vee r) \rightarrow (p \wedge \neg r)$
T	T	T	T	T	F	F
T	T	F	F	T	T	T
T	F	T	T	T	F	F
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	T	F	F	T	T	F
F	F	T	T	T	F	F
F	F	F	F	F	T	T

P and Q are only the statement variable that occur in this formula.

The truth table $(p \wedge (p \rightarrow q)) \rightarrow q$.

P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

The formula is tautology.

Truth table for $(p \vee Q) \rightarrow p$

P	Q	$p \vee Q$	$(p \vee Q) \rightarrow p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since, the truth table of $(p \vee Q) \rightarrow p$ Contains 'false'.

The formula is not tautology.

Q6. Write down the truth table for the following Compound statements and state which of them are tautology.

(i) $(q \vee r) \rightarrow (p \wedge r)$

(ii) $(p \wedge (p \rightarrow q)) \rightarrow q$.

→ The statements variable are p, q and r, 1st we Construct the truth table for this formula

Truth table for $(q \vee r) \rightarrow (p \wedge r)$.

p	q	r	$q \vee r$	$\neg r$	$p \wedge \neg r$	$(p \vee r) \to (p \wedge \neg r)$
					F	F
T	T	F	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	T	T	T	F
F	T	F	F	T	F	F
F	F	T	T	T	T	F
F	F	F	F	F	F	T

The formula is not tautology

(ii) p and q are only the statement variables that occur in this formula

Truth table for $(p \wedge (p \leftrightarrow q))$

p	q	$p \leftrightarrow q$	$p \wedge (p \leftrightarrow q)$	$p \wedge (p \leftrightarrow q))$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

The formula is tautology

Exercise

Check if $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ is tautology

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$q \vee (p \wedge \neg q)$	$(\neg p \wedge \neg q)$	$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

The formula is tautology.

Tautological implications and Equivalance of formulas

Defn (tautological imply)

A formula ϕ is said to tautologically imply a formula ψ if $\phi \to \psi$ is a tautological we write $\phi \to \psi$.

Example 1:

$(p \to Q) \Rightarrow (\neg Q \to \neg p)$

$\longrightarrow$ we prove this by using the truth table for $(p \to Q) \to (\neg Q \to \neg p)$.

p	Q	$\neg p$	$\neg Q$	$p \to Q$	$\neg Q \to \neg p$	$(p \to Q) \to (\neg Q \to \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$(p \to Q) \Rightarrow (\neg Q \to \neg p)$ is a tautology

Since all the entries in the last Column are 'true'

Hence $(p \to Q) \Rightarrow (\neg Q \to \neg p)$.

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Ex 2

$(p \rightarrow (Q \rightarrow R)) \Rightarrow ((p \rightarrow Q) \rightarrow (p \rightarrow R))$

Construct the truth table for

$(p \rightarrow (Q \rightarrow R)) \Rightarrow ((p \rightarrow Q) \rightarrow (p \rightarrow R))$

P	Q	R	$p \rightarrow Q$	$Q \rightarrow R$	$p \rightarrow R$	$p \rightarrow (Q \rightarrow R)$	$(p \rightarrow Q) \rightarrow (p \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

As the Columns of $p \rightarrow (Q \rightarrow R)$ and $(p \rightarrow Q) \rightarrow (p \rightarrow R)$ are identical

$\therefore (p \rightarrow (Q \rightarrow R)) \rightarrow ((p \rightarrow Q) \rightarrow (p \rightarrow R))$ is a tautology.

$\therefore (p \rightarrow (Q \rightarrow R)) \Rightarrow ((p \rightarrow Q) \rightarrow (p \rightarrow R))$.

Defn: Two formulas φ and ψ are said to be equivalent to each other if and only if $\varphi \leftrightarrow \psi$ and $\psi \Rightarrow \varphi$.

If $\varphi \leftrightarrow \psi$ is a tautology, we write $\varphi \Leftrightarrow \psi$.

pb1 $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$

The truth table for $p \vee q \leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$	$(p \vee q) \leftrightarrow \neg(\neg p \wedge \neg q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $(p \vee q) \leftrightarrow \neg(\neg p \wedge \neg q)$ is a tautology.

$p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

pb2 Show that $Q \Rightarrow (p \rightarrow Q)$

Construct the truth table for

$Q \rightarrow (p \rightarrow Q)$

P	Q	$p \to Q$	$Q \to (p \to Q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

$Q \longrightarrow (p \to Q)$

i) $(p \wedge Q) \longrightarrow (p \longrightarrow Q)$

ii) $\neg Q \wedge (p \longrightarrow Q) \longrightarrow \neg p$

iii) Show that $\neg\neg p \Longleftrightarrow p$

i) Construct the truth table for $(p \wedge Q) \Rightarrow (p \to Q)$

P	Q	$p \wedge Q$	$p \to Q$	$(p \wedge Q) \Rightarrow (p \to Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

ii) Construct the truth table for
$\neg Q \wedge (p \to Q) \to \neg p$

p	q	$\neg p$	$\neg Q$	$p \to Q$	$\neg Q \wedge (p \to Q)$	$\neg Q \wedge (p \to Q) \to \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Unit - 2

Normal forms

Definition

A product of (statement) variables and their negations is called an elementary product. Similarly, a sum of the variables and their negations is called an elementary sum.

The formulas $p, \neg p, \neg p \wedge q, \neg p \wedge \neg q$ are examples for elementary products. The formulas $p, \neg p, p \vee q, p \vee \neg q, \neg p \vee \neg q$ are examples for elementary sums.

Ex: Obtain a disjunctive normal form of

$p \to ((p \to q) \wedge \neg(\neg q \vee \neg p))$

$p \to ((p \to q) \wedge \neg(\neg q \vee \neg p))$

$\Rightarrow \neg p \vee ((p \to q) \wedge \neg(\neg q \vee \neg p))$

$\Rightarrow \neg p \vee ((\neg p \vee q) \wedge \neg(\neg q \vee \neg p))$

$\Rightarrow \neg p \vee ((\neg p \vee q) \wedge (q \wedge p))$

$\Rightarrow \neg p \vee (\neg p \wedge (q \wedge p)) \vee (q \wedge (q \wedge p)))$

$\Rightarrow \neg p \vee ((p \wedge \neg p \vee q)) \vee (q \wedge p)$

$\Rightarrow \neg p \vee p \vee (p \wedge q)$

$= \neg p \vee (p \wedge q)$

Ex. Obtain a disjunctive normal form of $\neg(p\vee q)\to(p\wedge q)$

$\neg(p\vee q)\to(p\wedge q)$

$\Rightarrow (\neg(p\vee q)\wedge(p\wedge q))\vee(\neg(\neg(p\vee q))\wedge\neg(p\wedge q))$

$\Rightarrow ((\neg p\wedge\neg q)\wedge(p\wedge q))\vee((p\vee q)\wedge(\neg p\vee\neg q))$

$\Rightarrow ((\neg p\wedge\neg q)\wedge(p\wedge q))\vee((p\vee q)\wedge\neg p)\vee((p\vee q)\wedge\neg q))$

$\Rightarrow (p\wedge\neg p\wedge q\wedge\neg q)\vee((p\vee q)\wedge\neg p)\vee((p\vee q)\wedge\neg q))$

$\Rightarrow F\vee(p\wedge\neg p)\vee(q\wedge\neg p)\vee(p\wedge\neg q)\vee(q\wedge\neg q)$

$\Rightarrow F\vee F\vee(q\wedge\neg p)\vee(p\wedge\neg q)\vee F$

$\Rightarrow (p\wedge\neg q)\vee(\neg p\wedge q)$

Ex. Find a disjunctive normal form of $(p\wedge\neg(q\vee r))\vee(((p\wedge q)\vee\neg r)\wedge p)$

$(p\wedge\neg(q\vee r))\vee(((p\wedge q)\vee\neg r)\wedge p)$

$\Rightarrow (p\wedge(\neg q\wedge\neg r))\vee(((p\wedge q)\vee\neg r)\wedge p)$

$\Rightarrow (p\wedge(\neg q\wedge\neg r))\vee(p\wedge q\wedge p)\vee(\neg r\wedge p)$

$\Rightarrow (p\wedge\neg q\wedge\neg r))\vee(p\wedge q)\vee(\neg r\wedge p)$

$\Rightarrow (p\wedge\neg q\wedge\neg r))$

Ex. Obtain a conjunctive normal form of $\cancel{\neg(p\vee q)\leftrightarrow(p\wedge q)}$ $(q\vee(p\wedge r))\wedge\neg((p\vee r)\wedge q)$

$(q\vee(p\wedge r))\wedge\neg((p\vee r)\wedge q)$

$\Rightarrow (q\vee(p\wedge r))\wedge(\neg(p\vee r)\vee\neg q)$

$\Rightarrow (q\vee(p\wedge r))\wedge((\neg p\wedge\neg r)\vee\neg q)$

$\Rightarrow (q\vee(p\wedge r))\wedge((\neg p\wedge\neg r)\vee(q\wedge\neg q)\vee((p\wedge r)\wedge(\neg p\wedge\neg r))\vee((p\wedge r)\wedge\neg q)$

$\Rightarrow (q\wedge(\neg p\wedge\neg r))\vee(q\wedge\neg q)\vee((p\wedge r)\wedge(\neg p\wedge\neg r))\vee((p\wedge r)\wedge\neg q)$

$\Rightarrow (\neg p\wedge q\wedge\neg r)\vee F\vee F\vee(p\wedge\neg q\wedge r)$

$\Rightarrow (\neg p\wedge q\wedge\neg r)\vee(p\wedge\neg q\wedge r).$

Q. Obtain the principle disjunctive normal form and principal conjunctive normal form of $P \leftrightarrow Q$

$\rightarrow$ Construct the truth table for $P \leftrightarrow Q$

P	Q	minterm	$P \leftrightarrow Q$
T	T	$P \wedge Q$	T
T	F	$P \wedge \neg Q$	F
F	T	$\neg P \wedge Q$	F
F	F	$\neg P \wedge \neg Q$	T

The principal disjunctive normal form of $P \leftrightarrow Q$ is

$(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The principal disjunctive normal form of $\neg(P \leftrightarrow Q)$ is $\neg((P \wedge \neg Q) \vee (\neg P \wedge Q))$

The obtain the principal conjunctive normal form of

$P \leftrightarrow Q$, consider $\neg((P \wedge \neg Q) \vee (\neg P \wedge Q))$

$\neg((P \wedge \neg Q) \vee (\neg P \wedge Q)) = (\neg P \vee Q) \wedge (P \vee \neg Q)$

So $(P \vee \neg Q) \vee (\neg P \vee Q)$ is the principal conjunctive normal form for $P \leftrightarrow Q$.

1. Obtain the PDNF for $\neg P \vee Q$.

Construct the truth table for $\neg P \vee Q$.

P	Q	Minterm	$\neg P \vee Q$
T	T	$P \wedge Q$	T
T	F	$P \wedge \neg Q$	F
F	T	$\neg P \wedge Q$	T
F	F	$\neg P \wedge \neg Q$	T

The principal disjunctive normal form of $\neg P \vee Q$ is

$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

The formula $(\neg P \vee Q)$ itself is in the principal Conjunctive normal form

2. By not using the truth table directly, find PDNF for

a) $P \leftrightarrow Q$

$P \leftrightarrow Q$

$\Leftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q)$

$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge Q) \vee (P \wedge Q) \vee (\neg Q \wedge Q)$

$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$

$\Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the PDNF of $P \leftrightarrow Q$

Hence $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ & the PCNF of $P \leftrightarrow Q$.

$(\neg P \vee Q) \wedge (P \vee \neg Q)$

(b) $\neg P\vee Q$ itself the ~~PDN~~ PCN form of $\neg P\vee Q$.

$(\neg P\vee Q) \Rightarrow (\neg P\wedge T)\vee(T\wedge Q)$

$\neg(P\vee Q) \Rightarrow (\neg P\wedge(Q\vee\neg Q))\vee((P\vee\neg P)\wedge Q)$

$\Rightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(P\wedge Q)\vee(\neg P\wedge Q)$

$\Rightarrow (P\wedge Q)\vee(\neg P\wedge Q)\vee(\neg P\vee Q)$

is the principal disjunction normal forms of $\neg P\vee Q$.

Theory of inference

If an implication $A\to B$ is a tautology, where A and B are statement formulae, we say that B logically follows from A or B is a valid conclusion of the premise of the hypothesis. we say that from set of premises $\{H_1, H_2, \dots H_n\}$ a conclusion C follows logically if $\{H_1\wedge H_2\wedge\dots\wedge H_n\} \Rightarrow C$

when C logically follows from $H_1, H_2 \dots H_n$

$$\begin{array}{l} H_1 \\ H_2 \\ \vdots \\ H_n \\ \hline C \end{array}$$

or $(H_1, H_2, \dots H_n \Rightarrow C)$ or (If $H_1, H_2, \dots H_n$ then)

Then means that if we know H_1 is true, H_2 is true, and H_n is true, then we can conclude that S is true.

Rule P: We may introduce a premise at any point in a derivation.

Rule T: We may introduce a formula S in a derivation if S is tautologically implied by any one or more of the preceding formulas in the derivation.

Ex: Demonstrate that S is a valid inference from the premises

$P \to \neg Q,\ Q \vee R,\ \neg S \to P$ and $\neg R$.

{1}	(1)	$Q \vee R$
{2}	(2)	$\neg R$
{1, 2}	(3)	Q
{4}	(4)	$P \to \neg Q$
{1, 2, 4}	(5)	$\neg P$
{6}	(6)	$\neg S \to P$
{1, 2, 4, 6}	(7)	S

Q. Show that $R \vee S$ is a valid conclusion from the premises $C \vee D$, $C \vee D \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$

[1]	(1)	$C \vee D$
[2]	(2)	$(C \vee D) \to \neg H$
[1, 2]	(3)	$\neg H$
[4]	(4)	$\neg H \to (A \wedge \neg B)$
[1, 2, 4]	(5)	$A \wedge \neg B$
[6]	(6)	$(A \wedge \neg B) \to (R \vee S)$
[1, 2, 4, 6]	(7)	$R \vee S$

Q. Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$, and Q.

[1]	(1)	$\neg R \vee P$
[2]	(2)	R
[1, 2]	(3)	P
[4]	(4)	$P \to (Q \to S)$
[1, 2, 4]	(5)	$Q \to S$
[6]	(6)	Q
[1, 2, 4, 6]	(7)	S
[1, 4, 6]	(8)	$R \to S$

Show that $P \to S$ can be derived from the premises
$\neg P \vee Q, \neg Q \vee R, R \to S$

[1]	(1)	$\neg P \vee Q$
[2]	(2)	P
[1,2]	(3)	Q
[4]	(4)	$\neg Q \vee R$
[1,2,4]	(5)	R
[6]	(6)	$R \to S$
[1,2,4,6]	(7)	S
[1,4,6]	(8)	$P \to S$.

Derive $P \to (Q \to S)$ using the rule CP if necessary from $P \to (Q \to R)$, $Q \to (R \to S)$

[1]	(1)	$P \to (Q \to R)$
[2]	(2)	P
[1,2]	(3)	$Q \to R$
[4]	(4)	$Q \to (R \to S)$
[1,2]	(5)	$\neg Q \vee R$
[4]	(6)	$\neg Q \vee (R \to S)$
[1,2,4]	(7)	$\neg Q \vee (R \wedge (R \to S))$
[1,2,4]	(8)	$\neg Q \vee S$

*. Construct the truth table for show that $\neg\neg P \Leftrightarrow P$

P	$\neg P$	$\neg\neg P$	$\neg\neg P \Leftrightarrow P$
T	F	T	T
F	T	F	T

$\neg\neg P \Leftrightarrow P$

Unit - 3

Def (Graph)

A Graph G is an order triple $(V(G), E(G), \varphi)$ consisting of a non-empty finite set $V(G)$; a finite set (may be empty) $E(G)$ which is disjoint from $V(G)$, and an incidence function φ that associates with each elements of $E(G)$ an unordered pair of elements of $V(G)$. The elements of $V(G)$ are called the vertices of G and the elements $E(G)$ are called the edges of G. If e is an edges and $\varphi(e) = (u, v)$, then we say that e is an edge joining u and v and the vertices u and v are called the ends of e.

$G = (V(G), E(G), \psi)$

where $V(G) = \{1,2,3,4,5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

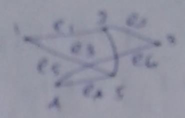
and ψ is defined by

$\psi(e_1) = (1,2)$, $\psi(e_2) = (2,3)$

$\psi(e_3) = (3,4)$, $\psi(e_4) = (4,5)$

$\psi(e_5) = (5,1)$, $\psi(e_6) = (5,2)$

(G)

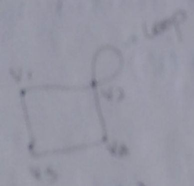


Defn: (Loop)

An edge of a graph that joins a vertex to itself is called loop or self loop.

(ex)

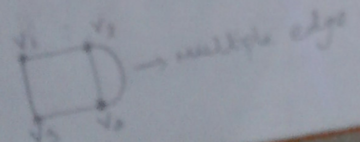
(G)



Defn: (multiple graph)

If more than one edge joining two vertices are allowed, then the resulting graph is called a multi graph (or) multiple graph.

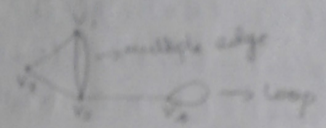
(ex) (G)



Defn : (pseudo graph)

If loops and multiple edges are allowed in a graph. Then the resulting graph is called pseudo graph.

(ex)

(a) 

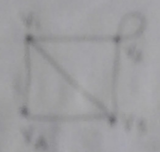
Defn : (Simple graph).

A graph which has no loops and no multiple edges is said to be a simple graph.

Defn : (Degrees).

The degree of a vertex v_i in a graph G is the number of edges incident with v_i. The degree of v_i is denoted by ~~d~~ $deg(v_i)$ (or) $deg(v_i)$ or, $d(v_i)$.

(ex).



$d(v_1) = 3$

$d(v_2) = 4$

$d(v_3) = 3$

$d(v_4) = 2$

maximum degree $\Delta(G) = \max\{deg\ v / v \in V(G)\}$

$= 4$

minimum degree $\delta(G) = \min\{deg\ v / v \in V(G)\}$

$= 2$

Th 1. The sum of the degrees of the vertices of a graph G is twice the number of degrees

(ie) $\sum \deg v = 2q$

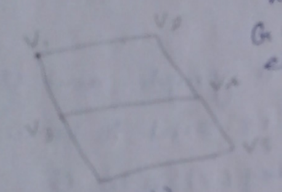
proof:

each edge has two end points so each edge will give degree two for both end points of it.

If there are q number of edges in a graph G, then the sum of degrees of the vertices of a graph G is equal to 2q.

see the following example

(G)



number of edges in G = 6 sum of all edge of all degree vertices = 10

Defn: (Regular graph)

If all the vertices of G have the same degree r then $\delta(G) = \Delta(G) = r$ regular graph of degree r.

Th 2. The number of vertices of odd degree in a graph is always even

→ let G be a graph with n vertices among those n vertices some of the vertices have odd degree and others have even degree.

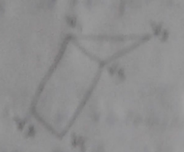
$\sum_{i=1}^{n} d(v_i) = \sum_{odd} d(v_i) + \sum_{even} d(v_i)$

$= 2$(no of edges of G) = even number

$=$ even

Hence, sum of vertices having odd degree is even.

Illustration is given below:



V_1 and V_2 have odd degrees whose sum is 6.

Defn:

* Let G be a graph and v be a vertex in G then (i) v is said to be an isolated vertex if $\deg(v) = 0$ in G

* v is called a pendent vertex if $\deg(v) = 1$ in G.

* G is said to be null graph if $E(G) = \phi$, when $E(G)$ is the edge set of G.

Defn (Sub graph)

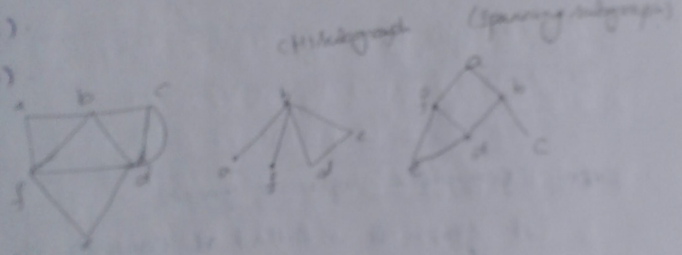
A graph $H = (V(H), E(H), \psi_H)$ is said to be subgraph of a graph $G = (V(G), E(G), \psi_G)$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the map ψ_H is the restricted by ψ_G to $E(H)$. If H is a subgraph of G, we write $H \subseteq G$.

Defn (Spanning sub graph)

A subgraph H of a graph G is said to be a spanning subgraph of G, if $V(H) = V(G)$.

(ex)

(G)



Defn Complete graph

A graph in which any two distinct vertices are adjacent is called a Complete graph. The Complete graph with p vertices is denoted by k_p

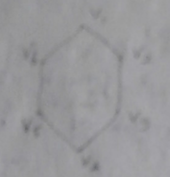
(ex)

Defn. (Bi-graph (or) Bipartite graph)

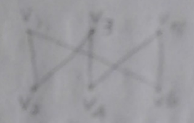
A graph G is called a bi-graph or bi-partite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. (V_1, V_2) is called a bi-partite of G.

Ex

G



G

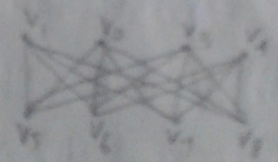


Defn. (Complete bipartite graph)

A graph G is called a complete bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. It is denoted by $K_{m,n}$. Here V_1 contains m vertices and V_2 contains n vertices.

Ex:

$K_{4,5}$



Defn. (Isomorphic)

Two graphs $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ are said to be Isomorphic to each other. If there is bijection of $V(G) \to V(H)$ and $\phi: E(G) \to E(H)$ such that $\psi_G = (u, v)$

If $\psi_H(\phi(e)) = (\phi(u), \phi(v))$.

If the graphs G and H Isomorphic, then we write $G \cong H$.

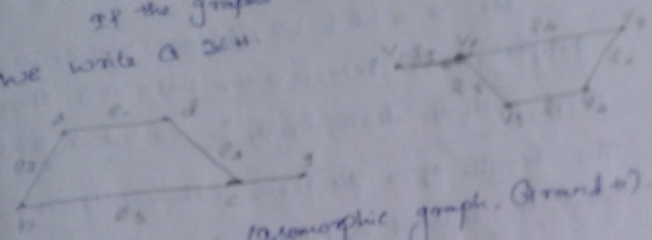


Figure: Isomorphic graph. (G and H).

paths. Cycles and Connectedness

First we Consider an undirected graph $G = (V, E)$.

A walk in G is a finite non-particle Sequence $W = v_0 e_1 v_1 e_2 v_2 e_3 \dots e_k v_k$ whose terms are alternatively vertices and edges, such that for $1 \le i \le k$, the vertices v_{i-1} and v_i are ends of the edge e_i. If $W = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ is a walk, it is said to be a walk form the vertex v_0 to the vertex v_k (or) a $(v_0 - v_k)$ walk. The Vertices v_0 and v_k are Called the origin and terminus of w.

Defn (path)

If $w: v_0 e_1 v_1 e_2 \ldots e_k v_k$ be distinct then the walk is called a path

(ie) $(v_0 - v_k)$ path.

Defn (Distance)

Let u and v be distinct vertices in a graph G. If there is a u-v path in G, then a u-v path of least length is called a geodesic. The length of a geodesic between u and v and its denoted by (u,v).

If there is no $(u-v)$ path in G. Then the $d(u,v) = \infty$

Defn (Connected).

Let G be a graph two vertices u and v of G are said to be Connected if either $u=v$ or there is a u-v path in G.

Theorem 1

A graph G is disconnected iff its vertex set V can be partitioned into two non-empty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex in V_1 and the other in V_2.

proof:

Assume that G is disconnected. Consider a vertex $u \in V$. Let $V_1 \{w - v\}$ there is a u-v path in G. As G is disconnected $V_1 \neq V$. Let $V_2 = V - V_1$, then $V_2 \neq \phi$ and $V_1 \cap V_2 = \phi$.

We claim that there is no edge with one end in V_1 and the other in V_2. If it is not so, let $e = ab$ be an edge in G such that $a \in V_1$ and $b \in V_2$. As $a \in V_1$ either $a = u$ or there is a $(u-a)$ path in G.

If $a = u$ then $e = ab = ub$ and $b \in V_1$ which is contradiction.

So if u, let $ux_1 x_2 \ldots x_m a$ be a $(a-a)$ path in G.

Clearly as $x_1, x_2 \ldots x_m$ are in V_1 b / x_i for only i and hence $ux_1 x_2 \ldots x_m$ ab in u-b path in G which is contradiction as $b \in V_1$. There is no edge e in G with one end of e in V_1 and the other end in V_2. Conversely, assume that the vertex set V can be partitioned into two disjoint non-empty subsets V_1 and V_2 such that these exists an edge in G.

where one end is in V_1 and the other in V_2. Consider a vertex $u \in V_1$ and a vertex $v \in V_2$. We claim that there is no u-v path $x_1, x_2, \dots, x_m, x_{m+1}$ in G.

Let i be the least positive integer such that $x_i \in V_2$ ($i \geq 1$). Since $x_0 = u \notin V_2$.

Also $x_{i-1} \in V_1$. Thus there is an edge with x_{i-1} and x_i as its end vertices and $x_{i-1} \in V_1$ and $x_i \in V_2$.

$\Rightarrow$ there is no u-v path in G.

Hence G is not connected.

Theorem: Let G be an undirected graph. Then G is bipartite iff it contains no odd cycle.

proof

Necessity

Let G be bipartite with bipartition (X, Y). Let $C: V_0 V_1 \dots V_k$ where $V_k = V_0$ be a cycle in G.

We may assume that $V_0 \in X$.

Then as $V_0 V_1$ is an edge and G is

Let G be bipartite with bipartition (X,Y). Let $C = v_0 v_1 \ldots v_k v_0$ where $v_k = v_0$ be a cycle. Since we may assume that $v_0 \in X$ then as $v_0 v_1$ is an edge and G is bipartite $v_1 \in Y$. As $v_1 \in Y$ and $v_1 v_2$ is an edge, it follows that $v_2 \in X$. proceeding like this, we have $v_{2i} \in X$. even so C is an even cycle. Thus G contains no odd cycle.

Sufficiency:

It suffices to prove the converse for connected graphs. Let G be a connected graph that contains no odd cycle. Choose an arbitrary vertex u and define a partition (X, Y) of V by defining.

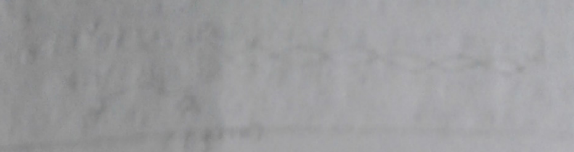
$X = \{v \in V \mid d(u,v)$ is even$\}$

$Y = \{v \in V \mid d(u,v)$ is odd$\}$

We claim that G is bipartite graph with bipartition (X,Y). Let v and $w \in X$. Let P be a shortest u-v path and Q be a shortest u-w path in G. Let u_1 be the last vertex common to P and Q. (u, u_1) sections of P and Q are shortest u-u_1 path and therefore have that same length say k. As the lengths of the paths P and Q are even, the lengths of (u_1, v) section P_1 of P and (u_1, w) section Q_1 of Q are both either even or odd.

If vw were an edge in G, then the cycle $P_1^{-1} Q_1 v$ is cycle of odd length.

Contrary to the hypothesis. Hence no two vertices in x are adjacent. Similarly no two vertices in y are adjacent. Thus (x, y) is a bipartition of the vertex set and G is bipartite.



Theorem 3: A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

proof:

We have the result by induction on the number of components of G. Let p(k) : If G is simple graph with k components, then it can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges, where $n = |V(G)|$

If $k=1$ then G is a simple connecting graph and hence the number of edges in G ≤ number of edges of K_n

$\leq \frac{(n-1)n}{2}$

where $n = |V(G)|$ and K_n is the complete graph of n vertices

Thus p(1) is true ⟶ 1

assume that p(m) is true for some m ⟶ 2

Let G be a simple graph with n vertices and (m+1) components. Let H_1 be a component of G

Let $|V(H_1)| = n_1$.

As G has m remaining components and each component has at least one vertex. We have $n_1 \le n-m$. Let H_2 be the subgraph of G induced by $V(G) - V(H_1)$. Then H_2 is a simple graph with $n-n_1$ vertices and m components and by (2) H_2 can have atmost $\frac{(n-n_1-m)(n-n_1-m+1)}{2}$ edges.

As H_1 is a connected simple graph with n_1 vertices by (1) it can have atmost $\frac{(n_1-1)n_1}{2}$ edges.

Thus the number of edges in G

$$G \le \frac{n_1(n_1-1)}{2} + \frac{(n-m-n_1)(n-m+1-n_1)}{2}$$

$$= \frac{1}{2}\left(n_1^2 - n_1 + (n-m)(n-m+1) + n_1(n-m+1+n_1-m+n_1^2)\right)$$

$$= \frac{1}{2}\left(n_1^2 - n_1 + (n-m)(n-m+1) - 2n_1(n-m) - 2n_1 + 2n_1^2\right)$$

$$= \frac{1}{2}\left[(n-m)(n-m-1+2) - 2n_1(n-m) + 2n_1(n_1-1)\right]$$

$$= \frac{1}{2}\left((n-m)(n-m-1) - 2(n-m)(n_1-1) + 2n_1(n-1)\right)$$

$$= \frac{1}{2}\left((n-m)(n-m-1)\right) \text{ as } n_1 \le n-m$$

$$\le (n-m)(n-m-1)$$ is also true.

From (1) & (2), $p(m+1)$ is also true.

By induction principal, $p(1)$ is true for all positive integers k.

UNIT-4

TREES

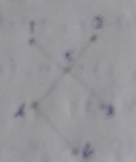
Defn (Tree)

A graph having no cycle is said to be acyclic. A tree is connected acyclic graph.

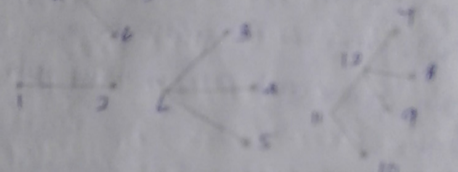
Defn (forest)

Collection of tree is called a forest. It is also an acyclic graph.

Ex:



(Tree)



(forest)

Theorem 1. A graph G without loops is a Tree iff any distinct vertices are connected by a unique path.

proof:

Let G be a tree and u and v be two distinct vertices of G. As G is connected there is at least one (u, V) path in G. is Connected there is at least one (u, V) path in G. Suppose that there are two distinct (u, v) path P_1 and P_2 in G. The path P_2^{-1} of P_2 is a closed in G. As every closed walk contains cycle this closed walk produces a cycle this cycle is a which is a contradiction to the fact that G is acyclic Thus there is one and only (u,v) path in G

Conversely,

Assume that there is exactly one path between any two vertices existence of a path between every pair of vertices appears that G is connected

If G contains a cycle, then there are at least two distinct vertices a, b on that cycle using the cycle we get (a,b) path in G which is a contradiction to assumption. So G is acyclic so G is connected and acyclic.

It is a tree

a tree with n vertices has $n-1$ edges

proof

We prove this result by strong induction in the number of vertices of G

Consider the statement $p(n)$

$p(n)$ a tree with n vertices has $n-1$ edges

If $n=1$, then the simple graph with only one vertex in K_1 which has $n-1 = 0$ edges so $p(1)$ is true. So assume that $p(n)$ is true for all $n < k$ for some $k \geq 1$ let G be a tree with $k+1$ vertices select an edge $e = uv \in E(G)$.

So G is a tree uv is the only (u,v) path in G and so there is no (u,v) path in $G-e$. Thus $G-e$

As G contained no cycle. there two component are also a cyclic and have they are tree with less than $k+1$ vertices. let the tree G_1 and G_2 contain n_1 and n_2 vertices respectively.

Then (i) $n_1 + n_2 = k+1 = n$.

(ii) as $n_1 \leq k$. and $n_2 \leq k$ by strong induction hypothesis G_1 and G_2 contains $n_1 - 1$ and $n_2 - 1$ edges so the number of edges in G = the no of edges in G_1 + there no of edges in G_2 + 1 (1 is for the edge $e = uv$)

$= n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1$

$= k + 1 - 1$

$= k$

Thus $|e(n)| = k$. so the statement $p(k+1)$ is also true. thus it $p(n)$ is true of all $n \leq k$. the $p(k+1)$ is also true. By the principal of strong induction $p(n)$ is true for all positive integer n.

Theorem : 3

A graph is a tree if it is minimally connected.

<u>Proof:</u>

let G be the tree. there G is Connected Suppose that G is not minimally Connected. than there is an edge $e = uv$ a Such that $G - e$ is

Connected. As $G-e$ is Connected, there is a (u,v) Path in $G-e$. So in G the path p together with these edge e will form a Cycle, which is Contradiction to the fact G is acyclic.

Conversely:

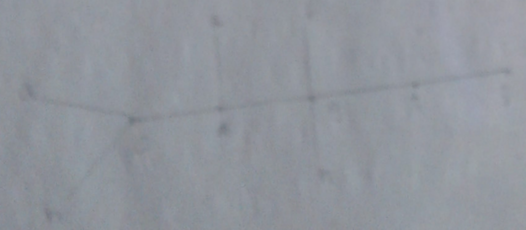
Assume that G is minimally Connected. So G is Connected. If there is a Cycle C in G. So One edge e in the cycle C then $G-e$ is also Connected.

→ Since G is minimally Connected. Thus G is Connected acyclic graph. (i.e) It is a tree.

Defn: (Center of a tree)

let T be a with n vertically. If u and v are distinct Vertical of T then $d(u,v)$ is the length of the unique (u,v) path. To Vertex v in T. the electricity $E(v)$ is defined as $E(v)$ = max (x, x_2) k center with minimum electricity is called the Center of T

Ex:



Defn: (Fundamental Circuits)

Let G be a Connected graph with n Vertices and T be a Spanning tree in G. If e=uv is an edge in G but not in T, then T+e Contains a cycle. Or there is a (u,v) path in T that Contains a cycle. The edge e Creates and additional Path between u and v in G and a Cycle in G. We observe that T+e Contains Only One Cycle. The unique Cycle in T+e is Called fundamental circuit of G. The edges of G which are not in T are called chords of T.

Defn: (Spanning Tree)

A Spanning Subgraph H of a graph is said to be Spanning tree if it is a tree.

Defn: (Complexity)

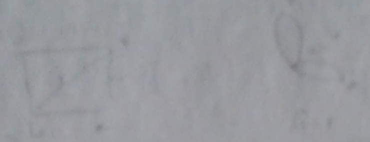
The no of Spanning tree in a Connected graph G is denoted by T(G) and is said to be the Complexity of graph G.

Defn: (Contraction)

If e=uv is an edge of G, then Contraction of e is the Operation of replacing U and V by a Single Vector whose incident edges are the edges others than e that were incident on u or v

The graph G.e may have edges

Ex:



Theorem:

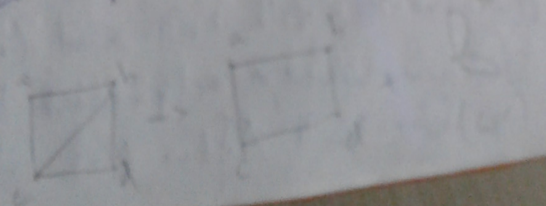
Let G be a Graph and $e \in E(G)$ then $\tau(G) = \tau(G-e) + \tau(G \cdot e)$

Proof:

The set S of spanning trees of G can be partitioned as $S_1 \cup S_2$ where S_1 is the set of the spanning trees of G not containing e and S_2 is the set of spanning trees of G containing the edge e. A spanning tree T of G does not contain e if and only if T is a spanning tree in $G-e$, we have $|S_1| = \tau(G-e)$

Ex: let us find $\tau(G)$ where G [illegible] edge

(a)



We Note that by omitting any one of the edges in $C_n - e$ we optional spanning tree for $G - e$ so $T(C_n e) = n$

Theorem-2 (Cayley's formula) If $n \geq 2$, $T(K_n) = n^{n-2}$

Proof
Let the vertices of the Complete Graph K_n be labeled with Integers 1, 2, ... n. We note that there are n^{n-2} sequence of length $n-2$ with entries from $\{1, 2, ... n\}$ Denote the set of all sequence by s. we establish a sequence Spanning trees of K_n and set S. Using the labels of the Vertices we Consider the vertex set $V(K_n)$ as an Ordered set To each spanning tree T we a sequence $f(T) = (t_1, t_2)$ as follows let s_1 be the 1st Vertex of degree one in T and t_1 the Unique Vertex which is adjacent to s_1 in T - s_1 and t_2 be the Unique Vertex which adjacent to s_2 in T-s let S_2 be the 1st Vertex of degree One in T - $\{s_1, s_2\}$ and t_2 be the Unique vertex adjacent to s_2 in T - $\{s_1, s_2\}$ After N-2 Iterations a

Single edge Remove

Thus we have produced a sequence $f(T) = (t_1, t_2, \ldots t_{n-2})$ of S. For tree T given in the following figure the associated sequence $f(T)(2, 3, 1, 3, 2, 1, 1)$

(A Spanning tree T and the associated sequence $f(T)$)

Next we define a function that produces a tree from each sequence x. Let $x = (t_1, t_2, \ldots t_{n-2})$ be the given sequence. First we begin with n vertices (and edge set $E = \phi$). Let s_1 be the least Number that does not appear in $(t_1, t_2, \ldots t_n)$. Join the vertices s_1 and t_1 by an edge and set $= \{s_1 t_1\}$ let $s_2 \neq s_1$ be the least Number that is Not in (t_2, t_3). Join the vertices s_2 and t_2 by an edge and let $E = \{s_1 t_1, \ldots$ Number such that $s_3 \notin (s_2, s_2)$ and s_3 is not in $(t_3, t_4 \ldots t_n)$ Set $E = \{s_1 t_1, s_2 t_2, s_3, t_3\}$

Continue in this way with he get edge s_t, s_{n-1}, $s_{n-2}+n-2$ T is

Now obtained by add the edj

joining the Remaining two vertices of $\{1, 2, \ldots n\} = \{s_1, s_2 \ldots s_{n-2}\}$ The Resulting Graph is Denoted by $g(t_1, t_2 \ldots t_{n-2})$.

At each Step we add one edge there are $n-1$ edges Including one that added at the least step.

After the ith steps in the Construction of $g(x)$: we have $n-i$ Components

So After adding the edge at the last step, we get a Connected graph so $g(x)$ is a Connected graph with n vertices and hence it is a spanning tree.

Clearly $g(f(T)) = T$ for every spanning tree T and hence the map f and g are bijections. Thus there are $|A| = n^{n-2}$ spanning trees in K_n.

$T(K_n) = n^{n-2}$.

Minimum Spanning tree.

Defn: (weighted graph)

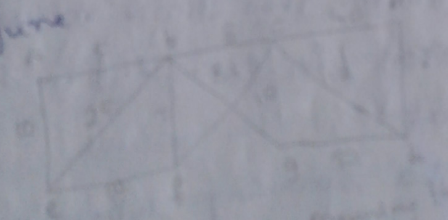
Let us consider simple connected graph G in which each edge e has been assigned some real number w(e) called a connected weighted graph.

To each spanning tree T of G we assign a weight by $w(T) = \sum_{e \in T}$ w(e) = sum of the weights of the edges of the spanning tree T.

Defn - (optional tree)

Amoung the collection of all spanning trees, we seek minimum weight spanning trees. A minimum weight spanning tree is called an option tree.

A weighted graph and spanning tree in it are shown in following figure



Bold edge form a spanning tree

Kruskal's Algorithm

We construct a spanning sub graph H. At each iteration we start with $E(H) = \phi$ and n component of H (as $E(H) = \phi$) every vertex of G is a component of H. At each iteration we consider a cheapest edge not in E(H). If that edge joins two components of H, then that edge is included to E(H). Otherwise discard it and consider the next cheapest edge and so on. The algorithm stops after $(n-1)^{th}$ interaction as at that stage the spanning graph H is a tree.

Algorithm

1. Choose an edge e_1 such that $w(e_1)$ is as small as possible.

2. If the edges $e_1, e_2, \ldots e_i$ have been chosen, then choose an edge e_{i+1} from $E(G) / \{e_1, e_2 \ldots e_i\}$ in such way that

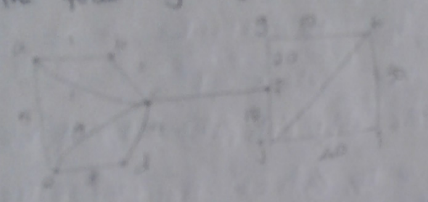
i) The spanning subgraph H whose $E(H) = \{e_1, e_2, \ldots e_i, e_{i+1}\}$ is acyclic.

(ii) $w(e_{i+1})$ is as small as possible subject to (i)

3. Stop when step 2 cannot be implemented further.

The spanning graph T produced by Kruskal's algorithm is optimal.

Example: Consider the connected weighted graph given in the following figure



Illustration

1. Select an edge with least weight. There are three edges least weight is one. Thus are three edges with weight one select anyone of these three edges. Let us select the edge ab.

2. Select an edge other than the edge ab whose weight is as small as possible. We select the edge ac.

3. Now the edge bc has weight one. But it should not be selected as this edge bc along with already selected edges ab and ac form a cycle. So next best edge is ac, whose weight is 5.

4. An edge other than ab, ac, bc and ac with least weight is edge. The edge ed along with already selected edges ab, ac, ae will not give any cycle. So at this iteration we select ed.

5. The next best edge which will not produce a cycle with (ab, ac, ae, ed) is either gh or jf. We select jf.

6. We select gh.

7. We select gf.

8. The edge jh should not be selected. We have to select the edge hc.

9. Select the edge cf.

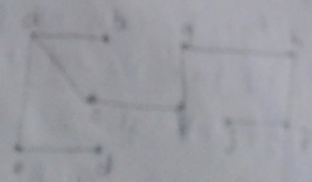
Algorithm stops.

The weight of this spanning tree

= $\sum$ cost (e_i)

= 1 + 1 + 5 + 8 + 10 + 10 + 20 + 20 + 40

= 125.



Draw n isolated vertices and label them $v_1, v_2, \ldots, v_n$. Tabulate the given weights of the edges of G in an n×n matrix. If $v_i v_j$ is not an edge, set the $(i,j)^{th}$ entry $= \infty$. Start from a vertex v_1 and connect it to its nearest neighbour say v_k. Now the edge $v_1 v_k$ is selected for our final tree. Consider v_1 and v_k as one subgraph and connected this subgraph to a vertex other that v_1 and v_k that has smallest entry among the entries of rows and row k. Let the new vertex be v_i. Next regard v_1, v_k and v_i as one subgraph and Continue the process until all n vertices have been connected by n-1 edges.

Ex(i) Consider the weighted graph in the following figure (a). The weight of the edges are isolated as in figure (b).

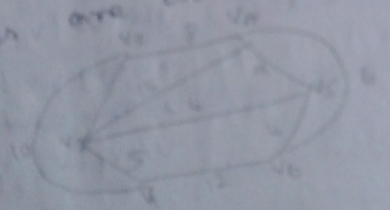


Figure (b)

$$
\begin{array}{c|cccccc}
 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\
\hline
V_1 & - & 5 & 10 & \infty & \infty & 12 \\
V_2 & 5 & - & 6 & 15 & 9 & \infty \\
V_3 & 10 & 6 & - & 8 & \infty & \infty \\
V_4 & \infty & 15 & 8 & - & 9 & 6 \\
V_5 & \infty & 9 & \infty & 9 & - & 4 \\
V_6 & 12 & \infty & \infty & 6 & 4 & -
\end{array}
$$

Iteration	Select the edge	Reason
1	$V_1 V_2$	wt $(V_1 V_2)$ is smallest in the I R(1)
2	$V_2 V_5$	In I and II rows (other than wt $(V_1 V_2)$)
3	$V_5 V_6$	In rows I, II and V but (V_5, V_6) is least and $V_6 \neq V_1, V_2 V_5$
4	$V_2 V_3$	as $V_3 \neq V_1, V_2, V_5, V_6$ and wt $(V_2 V_3)$ is min in I, II, V and VI rows
5	$V_6 V_4$	as wt $(V_6 V_4)$ is least among wt $(V_i V_j)$ [illegible] 1, 2, 5, 6 and $V_4 \neq V_1, V_2, V_3, V_5, V_6$

Dijkstra's Algorithm

Input: A weighted graph G and a starting vertex G. The weight of an edge xy is $w(x,y)$. Let $w(x,y) = \infty$ if xy is not edge in G.

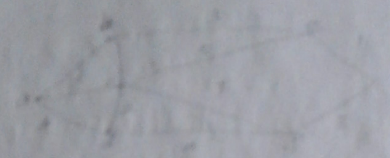
1. Let $S = \{u\}$, $d(u,v) = 0$, $t(x) = w(u,x)$ $\forall x \neq u$

2. For each $z \notin S$ for which v_z is an edge $t(z) \rightarrow \min\{t(z), t(v) + w(vz)\}$

3. If $S = V(G)$, stop. If $S \neq V(G)$ go to step (2)

When the algorithm terminates, weight of a shortest (u, v) path is given by the final value of $t(v)$ $\forall v \neq u$.

We illustrate the algorithm by considering the weighted graph given in the following figure (a).



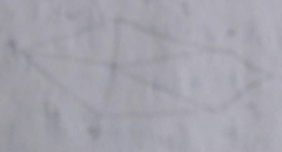
(figure 2)

Let us find the shortest path from the vertex a

Iteration.

S $t(x)$: $\{2, \emptyset\}$ and

1. $\{a\}$ $t(b)=2, t(c)=1, t(a)=0$
2. $\{a, c\}$ $t(b)=2, t(d)=3, t(x)=6$
3. $\{a, c, b\}$ $t(d)=3, t(e)=5, t(x)=$
4. $\{a, c, b, d\}$ $t(e)=5, (z)=7$
5. $\{a, c, b, d, e\}$ $t(f)=7, t(z)=6$
6. $\{a, c, b, d, e, z\}$ $t(f)=7$
7. $\{a, c, b, d, e, z, f\}$.

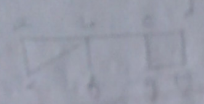


Cut Sets and Cut vertices

Defn: (cut set).

In a connected graph G, a cut set is a set of edges whose removal from G leaves G disconnected provided from no paper subset of those edges disconnects G

Ex:



Theorem

Every cut set in a connected graph G that contain atleast one edge it every spanning three of G

Proof

Let S be a cut set in G and T be a spanning time in G A T is a tree in connected. If S does not contain every of the edges of T then for the graph G-S the graph obtained after the removal of S from G. T will remain as a connecting spanning subgraph and hence G-S is connected.

⟹ to the fact that S is a subset

Thus S contain atleast one edge of T

Theorem:

In a connected graph G any minimal set of edges containing at least one edge of every spanning tree of G is a cut set

Proof:

Let S be a minimal set of edge containing atleast one edge of every spanning tree of G so G-S will contain any spanning tree and hence it is disconnected. If A is a proper subset of the set S, there is a spanning tree T of G such that A contains no edge of T. Then G-A contains T and hence G-A is connected

Thus G-S is disconnected but G-A is connected but cut-set in G

Theorem:

If S is a cut set of a connected graph G then G-S contains exactly two connected components.

Proof:

Let S be a cut set in a connected graph G. Then G-S is disconnected and

$G-A$ is not disconnected for every proper subset $A \subseteq S$. Let $C_1, C_2, \ldots, C_k$ be the components of G. We claim that $k = 2$. Assume that $k \geq 3$. As G connected there is an edge e in G which has one end in G for some $i = 1, 2, \ldots k$. Clearly $e \in S$. Take $A = S - \{e\}$. Then the components of $G - A$ are $(S_i \cup S_j)$ & S_j ($j \neq i, 1$) and hence $G-A$ is disconnected.

$\Rightarrow \Leftarrow$

Thus $k = 2$.

Defn: (Fundamental cutset)

A cutset S containing exactly one edge of a tree T is called a fundamental cutset or a basic cutset with respect to T.

Defn (Cut - vertex)

Let v be a vertex in a connected graph G. If $G - v$ is disconnected then v is said to be a cut - vertex of G.

Theorem Let v be a vertex in a connected graph G. then v is a cutset if and only if there exists two vertices x and y ($\neq v$) in G such that every path but x and y through v

$\longrightarrow$ Let v be cut-vertex. Then $G-v$ is disconnected. Select two vertices x and y in $G-v$ but that they lie in different components of $G-v$. As G is disconnected there are (x, y) paths in G. As x and y lie in different components of $G-v$ there is an (x, y) path in $G-v$. Thus every (x, y) path in G must pass through the vertex v.

Conversely assume that every (x, y) path in G passes through v. then there is an (x, y) path in $G-v$ and hence $G-v$ is disconnected. Therefore v is a cut-vertex to be a cut-vertex of G.

Corollary

In a tree T, a vertex is a cut-vertex cut-vertex ~~do~~ $\log(v)/1$

proof.

① $deg(v) \neq 1$. Select two vertices x and y such that x & y are edges in T. Then xvy is the unique (x, y) path in T and this path passes through the vertex v. So v is a cut-vertex.

Conversely if v is a cut vertex, there exist two vertices x and y in $T-v$ such that the (x, y) path in T passes through v. On this path we can find two vertices a, b such that the av, vb in T.

So $deg(v) \geq 2$.

② If v is cut vertex, the $G-v$ [illegible] more than two components way hence. For example let $G = K_{1,n}$ with $n \geq 2$ and v be the unique vertex of G with $deg(v) = n$.

Thus v is a cut-vertex in G and $G-v$ has n components.

Vertex connectivity

The vertex connectivity of a graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph either disconnected or a graph with [illegible] vertex the vertex connectivity of a graph G is said to be k connected if $k(G) \geq k$

Eg:
$k(G) = 0$, if G is disconnected or k_1

$k(G) = 1$, if G is a tree with more than one vertex.

$k(G) = n-1$, if $G = K_n$.

$k(G) = 2$ if $G = C_n$ a cycle on G n vertices when $n \geq 3$

Edge Connectivity:

A cut-set S is said to be a k edge cut if $|S| = k$. The edge connectivity $k'(G)$ of G is the minimum for which G has a k edge cut. G is said to be k-edge connected if $k'(G) \geq k$

Eg:
$k'(G) = 0$ if G is K_1 or disconnected

$k'(G) = 1$ if G is a tree with at least two vertices

$k'(G) = n-1$ if $G = K_n$

In the following graph,

$k(G) = 1$ and $k'(G) = 2$.

Theorem: For any graph G, $k \le k' \le \delta$

We, prove $k \le \delta$. If G has no edges. $k' = \delta = 0$. Otherwise removal of minimum degree results in a disconnected graph. Hence $k' \le \delta$.

Now, to prove $k \le k'$ we consider the following cases.

Case (i)

G is disconnected or trivial. Then $k = k' = 0$.

Case (ii)

G is a connected graph with a cut edge e. Then $k' = 1$. Further in this case $G = K_2$ or one of the vertices incident with e is a cut vertex. Hence $e = 1$ so that $k = k' = 1$

Case (iii)

$k' \ge 2$. Then there exists k' edges the removal of which disconnect the graph. Hence the removal of $k-1$ of these edges results in a graph G with a cut edge. For each of these $k'-1$ edges select an incident vertex different from u or v. The removal of these $k'-1$ points removal all the $k'-1$ edges. If the resulting graph is disconnected, then $k \le k'-1$. If not e is a cut edge of this subgraph and hence removal of u or v results in a disconnected or trivial graph.

Hence $k \le k'$ and complete the proof.

EULERIAN GRAPHS

Let us consider undirected G. A closed trial in G is a closed walk in which no edge of G is used more than once.

A tour of G is a closed walk that traverse each edge of G atleast once.

A graph G is said to be eulerian if it contains an euler tour.

Theorem: If a graph G contains no odd degree vertex and e is a edge in G then there is a closed trial in G containing e.

Let C be a longest trial in G containing the edge e. (If e = uv, then uv is itself a trial in G containing e so such a trial (exists) let C be $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ and $e = e_1$ for some) we claim that $v_0 = v_k$. As C is a longest trial containing e. all the edges in C of an edge $e' = v_k v$ is not included in C the trial $v_0 e_1 v_1 \dots e_k v_k e' v$ is longer than C) if $v_0 \neq v_k$ then $\deg(v_k) = 2m+1$ where m is the number of times v_k occurs as an internal vertex in C. This is a contradiction as degree of every vertex is even in G. Thus $v_0 = v_k$ and C is a closed trial containing e.

Thm: A given connected graph G is eulerian if and only if all the vertices of G are of even degree.

$\longrightarrow$ Let G be eulerian and C be an euler tour in G. let u be the starting and terminus vertex of C. In tracing the tour C from u every time the tour passes through an internal vertex v, two of the edges incident with v are accounted for (one edge to enter v and the other to leave from v). When it terminates at u it uses one of the edges incident with v. Thus as C contains each edge of G exactly one we have $deg(v) = 2k$ if $v \neq u$ and v appears k times as an internal vertex in C and $deg(u) = 2k + 2$ if u appears k times as an internal vertex in C.

Here degree of each vertex of G is even converse part.

Let G be a connected graph which contains no odd degree vertex. Let C be a longest trial in G

Unit-5

Lattices And boolean Algebras

One of the important concepts in mathematics is that of relation.

We recall that a relation R on a non empty set A is a subset of A x A if $(x,y) \in R$ then we write xRy. A Relation R on A is said to be reflexive if a Ra holds good of every element $a \in A$, Symmetric if for all a, b in A when ever $(a,b) \in R$, then $(b,a) \in R$. The Relation R is said to be antisymmetric if $(a,b) \in R$, then $(b,a) \notin R$ said to be transitive if whenever aRb and bRc, then Rc

* A Relation R is said to be a an equivalence Relation if R is Reflexive Symmetric and transitive

* A Relation R is said to be Partial ordering on A if R is Reflexive antisymmetric and transitive

* If R is a Partial Ordering on A if, then (A,R) is Called a Partially

Ordered set a poset usual we write $\leq$ or $\leq$ instead of R. [illegible]

HASSE DIAGRAMS

A Partially ordered finite set $(A, \leq)$ can be graphically Represented by diagram as points in the plan such that if $a, b \in A$ such that $a \leq b$ and $a \neq b$ exactly vertically above a; there may be a devitation to the left or Right of the vertical line through a. If $a < b$ and there is no $c \in A$ such that $a < c$ and $c > b$ then a and b are connected by a line segment for every the Hasse diagram of the poset $(P(X), \subseteq)$ is given below, where $X = \{1, 2, 3\}$ and $\subseteq$ is the relation 'subset of'

Definition 1

A partial order relation $\leq$ on a set A is called a total order or linear order. If for every $a, b \in A$ either $a \leq b$ or $b \leq a$. If $\leq$ is a total order on A then the poset $(A, \leq)$ is called or a totally ordered set

Definition: 2

Let $(x, \leq)$ be a poset and $a, b \in x$ If there is an element c in x such that $a \leq c$ and $b \leq c$ then c is said to be an bround (l.u.b) of a and b if

i) c is a lowerbound upper bound a and b (i.e. $a \leq c$ and $b \leq c$ and

ii) whenever d is an upper bound of a and b then $c \leq d$ (i.e. $a \leq d$ and $b \leq d \Rightarrow c \leq d$)

An element c is said to be a lower bound of a and b if $c \leq a$ and $c \leq b$ An element is said to be greatest lower bound (g.l.b) of a and b if

i) c is a lower bound of a and b and

ii) whoever d is a lower bound of a and b the $d \leq C$

Definition: 4

A poset $(X, \leq)$ is said to be a lattice if for every $a, b \in X$, both $a \vee b$ and $a \wedge b$ exist.

Theorem

Every chain is a lattice

proof

Let $(X, \leq)$ be a chain and $a, b \in X$.

Then we have either $a \leq b$ or $b \leq a$.

Case 1:

Assume that $a \leq b$. Clearly b is an upper bounded of a and b. If c is an upper bound of a and b, we have $a \leq c$ and $b \leq c$. Thus $b \leq c$, for every upper bound c of a and b. Hence b is the least upper bound of a and b.

Similarly $a \wedge b = a$.

Case 2:

Assume that $b \leq a$. Then we can prove that $a \vee b = a$ and $a \wedge b = b$.

Thus $a \vee b$, $a \wedge b$ exist for all $a, b \in X$

And hence $(X, \leq)$ is a lattice.

Definition 4

let $(X, \leq)$ be a poset. If there is a element $a \in X$ such that $a \leq x$, for all $x \in X$, then a is said to be a least element in X. (If there is a least element in $(X, \leq)$, then it is unique. The greatest element, if it exists is denoted by 1, and the least element, if it exists, is denoted by 0. A lattice which has both 0 and 1 is called a bounded lattice.

Some Properties of lattices

let $(L, \leq)$ be a lattice. Then L satisfies the following laws.

Idempotent laws:

$a \wedge a = a$ and $a \vee a = a$, for all $a \in L$.

Commutative laws:

$a \wedge b = b \wedge a$ and $a \vee b = b \vee a$, for all $a, b \in L$.

Associative laws:

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and

$(a \vee b) \vee c = a \vee (b \vee c)$ for all $a, b, c \in L$.

Absorption laws

$a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ for all $a, b \in L$

proof

let $a \in L$

Duality principle

let $(L, \leq)$ be a lattice. If we define a relation $\leq$ in L as follows: for all $a, b \in L$, $a \leq b$ if and only if $b \leq a$ in $(L, \leq)$, then $\leq$ is also a partial ordering & clearly for all $a, b \in L$.

lub $\{a, b\}$ in $(L, \leq)$ = glb $\{a, b\}$ in $(L, \leq)$ and glb $\{a, b\}$ in $(L, \leq)$ = lub (a, b) in $(L, \leq)$. This lattice is called the dual of the lattice $(L, \leq)$.

for example

In any lattice $(L, \leq)$ the statement $a \leq b \Rightarrow a \vee b = b$ is valid. Hence its dual statement $a \geq b \Rightarrow a \wedge b = b$ is valid.

Theorem

In any lattice $(L, \leq)$ the operations $\vee$ and $\wedge$ are isotones (ie) if $y \leq z$ in L then $x \wedge y \leq x \wedge z$ and $x \vee y \leq x \vee z$ for all $x \in L$.

Let $x, y, z \in L$ and $y \leq z$.

By idempotent law. $x \wedge x = x$

As $y \leq z$. we have $y \wedge z = y$.

$x \wedge y = x \wedge ((x \wedge y) \wedge z)$

$= x \wedge ((y \wedge x) \wedge z)$

$= x \wedge (y \wedge (x \wedge z))$

$= (x \wedge y) \wedge (x \wedge z)$

$\therefore \leq (x \wedge y) \wedge (x \wedge z)$.

Lattice through Algebraic Operations

We have already proved that if $(L, \leq)$ is a lattice and if the binary operation $\wedge$ and $\vee$ on L are defined by $a \wedge b = glb\{a, b\}$; $a \vee b = lub\{a, b\}$ for all $a, b \in L$, then the binary operations $\wedge$ and $\vee$ satisfy commutative law, associative law, absorption law and idempotent law. We now prove that if L is a non-empty set and $\sqcap$ and $\sqcup$ are

a

two binary operations on L satisfying these laws, then we can define a suitable order $\leq$ in L such that $(L, \leq)$ becomes a lattice and $a \wedge b$ = glb {a, b} and $a \vee b$ = lub {a, b} with respect to this order relation $\leq$.

New lattices defn:

A non-empty subset S of a lattice L is called a sublattice of L if S is closed under the operations 'join' and 'meet' of L.

(ii) for all $s_1, s_2 \in S$, the glb $\{s_1, s_2\}$ and lub $\{s_1, s_2\}$ in the lattice L are elements of S.

ex: let $(L, \leq)$ be a lattice.

(a) If $x \in L$, the set $\{x\}$ is a sublattice of L.

(b). If $x, y \in L$ such that $x \leq y$ then the set $[x, y] = \{z \in L \mid x \leq z \leq y$ in $L\}$ is a sublattice of L

Definition

Two posets $(P, \leq)$ and $(Q, \leq)$ are called order isomorphic if there is a bijective map $f: P \to Q$ such that $x \leq y$ in P if and only if $f(x) \leq f(y)$ in Q.

Theorem

Every meet-homomorphism is an order preserving map.

Proof.

Let $f: L_1 \to L_2$ be a meet-homomorphism from a lattice $(L_1, \leq)$ to a lattice $(L_2, \leq_2)$. Let $a \leq b$ in L_1. Then $a \wedge b = a$.

As f is a meet homomorphism we have $f(a) = f(a \wedge b) = f(a) \wedge f(b)$.

Thus $f(a) \wedge f(b) = f(a)$ in L_2. Hence $f(a) \leq_2 f(b)$ in L_2. Thus $a \leq b$ in $L_1 \Rightarrow f(a) \leq f(b)$ in L_2. And f is an order-preserving map.

Theorem

Let f be an (order) isomorphism from a poset $(L, \leq)$ onto a poset $(M, \leq')$. If L is a lattice then M is also a lattice and f is a lattice isomorphism.

Proof:

Let f be an (order) isomorphism from a poset $(L, \leq)$ onto a poset $(M, \leq)$. If L is a lattice then M is. f is a bijection there exist a unique $a \in L$ such that $f(a) = x$ and a $y \in M$ and f I such that $f(b) = y$. Let $c = a \vee b$). Let $z = f(c)$ in M. As $a \leq c$, $b \leq c$ and f is order preserving have $f(a) \leq f(c)$ and $f(b) \leq f(c)$ in M i.e. We have $x \leq z$ and $y \leq z$ in M and z is upper bound for $\{x, y\}$ in M. Let $w \in M$ be an upper bound for $\{x, y\}$ in M. Let d be the unique element in L such that $f(d) = w$. As $x \leq w$, $y \leq w$ (i.e. $f(a) \leq f(d)$, $f(b) \leq f(d)$) and f is an order isomorphism we have $a \leq d$ and $b \leq d$ in L so $c = a \vee b \leq d$ in L. As $c \leq d$ we have $f(c) \leq f(d)$ in M so we get $z \leq w$ in M. Thus we have proved that

i) z is upper bound $\{x, y\}$ in M.

ii) w is an upper bound for $\{x, y\}$ in M then $z \leq w$

In othe words we have proved that $x \vee y$ exists in M and $x \vee y = z = f(c) = f(a \vee b)$, where $f(a) = x$ and $f(b) = y$.

So M is a lattice and as the bijection $f : L \to M$ satisfies $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$, for all $a, b \in L$. f is a lattice isomorphism.

Product lattice of two lattices.

If L and M are lattices then we can give a lattice structure to the Cartesian product $L \times M$, using the partial order relations of L and M.

Let $(L, \wedge, \vee)$ and $(M, \sqcap, \sqcup)$ be two lattices. Consider the cartesian product $L \times M = \{(x, y) : x \in L$ and $y \in M\}$. We define binary operations $\wedge$ and $\vee$ on $L \times M$ as follows.

for all $(x_1, y_1), (x_2, y_2) \in L \times M$.

$(x_1, y_1) \wedge (x_2, y_2) \in L \times M$

$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \sqcap y_2)$ and

$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \sqcup y_2)$

$(x_1, y_1) \vee (x_2, y_2)$ ...

We now prove that $(L \times M, \wedge, \vee)$ is a lattice.

verifying the

$(L \times M, \wedge, \vee)$ is a lattice.

We now show that $\wedge$ and $\vee$ defined above satisfy

a) commutative law

b) associative laws and

c) absorption laws

As $(L, \wedge, \vee)$ and $(M, \sqcap, \sqcup)$ are lattices, the operations $\wedge$ and $\vee$ on L and the operations $\sqcap$ and $\sqcup$ on M satisfy these laws.

Let $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in L \times M$.

(a) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \sqcup y_2)$

$= (x_2 \vee x_1, y_2 \sqcup y_1)$

$= (x_2, y_2) \vee (x_1, y_1)$.

Similarly $(x_1, y_1) \wedge (x_2, y_2) = (x_2, y_2) \wedge (x_1, y_1)$

(b) $(x_1, y_1) \vee ((x_2, y_2) \vee (x_3, y_3)) = (x_1, y_1) \vee (x_2 \vee x_3, y_2 \sqcup y_3)$

$= (x_1 \vee (x_2 \vee x_3), y_1 \sqcup (y_2 \sqcup y_3))$

$= ((x_1 \vee x_2) \vee x_3, (y_1 \sqcup y_2) \sqcup y_3)$

$= (x_1 \vee x_2, y_1 \sqcup y_2) \vee (x_3, y_3)$

$= (x_1 \vee x_2, y_1 \sqcup y_2) \vee (x_3, y_3)$

$= ((x_1, y_1) \vee (x_2, y_2)) \vee (x_3, y_3)$.

Similarly $(x_1, y_1) \wedge ((x_2, y_2)) \wedge (x_3, y_3) =$

$\{((x_1, y_1) \wedge (x_2, y_2)) \wedge (x_3, y_3)$

(c) $(x_1, y_1) \vee ((x_1, y_1) \wedge (x_2, y_2)) = (x_1, y_1) \vee (x_1 \wedge x_2, y_1 \wedge y_2)$

$= (x_1 \vee (x_1 \wedge x_2), y_1 \vee (y_1 \wedge y_2))$

$= (x_1, y_1)$

Similarly $(x_1, y_1) \wedge ((x_1, y_1) \vee (x_2, y_2)) = (x_1, y_1)$

Thus $(L \times M, \wedge, \vee)$ is a lattice.

Boolean Algebras def:

A complemented distributive lattice is called Boolean Algebra or a boolean lattice.

Theorem: In a boolean algebra L, the De Morgan's law given by $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$ hold for all $a, b \in L$.

Let L be a boolean algebra and $a, b \in L$.

Then $(a \wedge b) \vee (a' \vee b') = (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b'))$

$= ((a \vee a') \vee b') \wedge ((b \vee b') \vee a')$

$= (1 \vee b') \wedge (1 \vee a')$

$= 1 \wedge 1$

$= 1$

~~$(a \wedge b) \wedge (a' \vee b')$~~ $= 1$

$(a \wedge b) \wedge (a' \vee b') = ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b')$

$= (a \wedge a') \wedge b) \vee (a \wedge (b \wedge b'))$

$= (0 \wedge b) \vee (a \wedge 0)$

$= 0 \vee 0 = 0$

Definition

Let a and b are two elements in a lattice. The element b is said to be a cover for a if $a < b$ and there is no element c in the lattice such that $a < c$ and $c < b$. If b covers the element a, we write $a \lessdot b$. An element which covers the element 0 is said to be an atom of the lattice.

Theorem

Let B be a finite boolean algebra. If $b \neq 0$ is an element in B, then there exists an atom a such that $a \leq b$.

$\longrightarrow$

If b itself is an atom, then we take $a = b$. If b is not an atom, as B is finite, we can find a chain $0 < b_n < b_{n-1} < \dots < b_1 < b$ satisfying $a \lessdot b_i$; $b_i < b_{i+1}$ for all $i = 1, 2, \dots, n-1$ and $b_n < b_1$ so b_n is an atom such that $b_n < b$ and we take $a = b_n$.

Definition

Let B_1 and B_2 be boolean algebras. A mapping $f: B_1 \to B_2$ is called a boolean homomorphism from B_1 into B_2 if f is a lattice homomorphism and for all $x \in B_1$ we have $f(x') = (f(x))'$. A boolean homomorphism is said to be an isomorphism, if it is bijective. If there is a boolean isomorphism between B_1 and B_2, we write $B_1 \cong B_2$.

Now we have the following representation theorem for finite boolean algebras.

Theorem: Let B be a finite boolean algebra and let A be the set of all atoms of B. Then the boolean algebra B is isomorphic to the boolean algebra $P(A)$.

proof

① h is one-to-one. If $h(b_1) = h(b_2)$ for $b_1, b_2 \in B$. Let $h(b_1) = h(b_2) = \{a_1, a_2 \ldots a_n\}$. Then by theorem 24 $b_1 = a_1 \vee a_2 \vee \ldots \vee a_n = b_2$.

So $h(b_1) = h(b_2) \Rightarrow b_1 = b_2$.

② h is into. Let $c \in P(A)$.

Thus if $a \in A(b)$, then $a \in c$ implying $A(b) \subseteq c$ and $b = c_1 \vee c_2 \vee \ldots \vee c_n$. Thus we have $c = A(b)$ and $c \leq A(b)$ W.h.f. c. So h is into.